

# On minimal generating systems for matrix $O(3)$ -invariants

A.A. Lopatin

*Fakultät für Mathematik, Universität Bielefeld, Postfach 100131, 33501 Bielefeld, Germany*

*Institute of Mathematics, SBRAS, Pevtsova street, 13, Omsk 644099, Russia*

*artem\_lopatin@yahoo.com*

## Abstract

The algebra of invariants of several  $3 \times 3$  matrices under the action of the orthogonal group by simultaneous conjugation is considered over a field of characteristic different from two. The maximal degree of elements of minimal system of generators is described with deviation 3.

2000 Mathematics Subject Classification: 16R30; 13A50.

Key words: invariant theory, classical linear groups, polynomial identities, generators.

## 1 Introduction

All vector spaces, algebras, and modules are over an infinite field  $\mathbb{F}$  of characteristic  $\text{char } \mathbb{F} \neq 2$ . By algebra we always mean an associative algebra.

The algebra of *matrix  $O(n)$ -invariants*  $R^{O(n)}$  is a subalgebra of the polynomial ring

$$R = R_n = \mathbb{F}[x_{ij}(k) \mid 1 \leq i, j \leq n, 1 \leq k \leq d],$$

generated by  $\sigma_t(A)$ , where  $1 \leq t \leq n$  and  $A$  ranges over all monomials in matrices  $X_1, \dots, X_d, X_1^T, \dots, X_d^T$ . Here  $\sigma_t(A)$  stands for  $t^{\text{th}}$  coefficient of the characteristic polynomial of  $A$  and

$$X_k = \begin{pmatrix} x_{11}(k) & \cdots & x_{1n}(k) \\ \vdots & & \vdots \\ x_{n1}(k) & \cdots & x_{nn}(k) \end{pmatrix}$$

is  $n \times n$  *generic* matrix, where  $1 \leq k \leq d$ . Moreover, we can assume that  $A$  is *primitive*, i.e., is not equal to the power of a shorter monomial. By the Hilbert–Nagata Theorem on invariants,  $R^{O(n)}$  is a finitely generated algebra, but the mentioned generating system is not finite.

Similarly to matrix  $O(n)$ -invariants we can define matrix  $GL(n)$ ,  $Sp(n)$  and  $SO(n)$ -invariants. Their generators were found in [16], [13], [1], [4], [18], and [9]. For  $\text{char } \mathbb{F} = 0$  relations between generators for  $GL(n)$ ,  $O(n)$ , and  $Sp(n)$ -invariants were computed in [15] and [13]. For any  $\text{char } \mathbb{F}$  relations for  $GL(n)$  and  $O(n)$ -invariants were established in [17], [11]. Note that in the case of  $O(n)$ -invariants we always assume  $\text{char } \mathbb{F} \neq 2$ .

For  $f \in R$  denote by  $\deg f$  its *degree* and by  $\text{mdeg } f$  its *multidegree*, i.e.,  $\text{mdeg } f = (t_1, \dots, t_d)$ , where  $t_k$  is the total degree of the polynomial  $f$  in  $x_{ij}(k)$ ,  $1 \leq i, j \leq n$ , and  $\deg f = t_1 + \dots + t_d$ .

Since  $\deg \sigma_t(Y_1 \cdots Y_s) = ts$ , where  $Y_k$  is a generic or a transpose generic matrix, the algebra  $R^{O(n)}$  has  $\mathbb{N}$ -grading by degrees and  $\mathbb{N}^d$ -grading by multidegrees, where  $\mathbb{N}$  stands for non-negative integers.

Given an  $\mathbb{N}$ -graded algebra  $\mathcal{A}$ , denote by  $\mathcal{A}^+$  the subalgebra generated by homogeneous elements of positive degree. A set  $\{a_i\} \subseteq \mathcal{A}$  is a minimal (by inclusion) homogeneous system of generators (m.h.s.g.) if and only if  $\{a_i\}$  are  $\mathbb{N}$ -homogeneous and  $\{\overline{a_i}\}$  is a basis of  $\overline{\mathcal{A}} = \mathcal{A}/(\mathcal{A}^+)^2$ . If we consider  $a \in \mathcal{A}$  as an element of  $\overline{\mathcal{A}}$ , then we usually omit the bar and write  $a \in \overline{\mathcal{A}}$  instead of  $\overline{a}$ . An element  $a \in \mathcal{A}$  is called *decomposable* if  $a = 0$  in  $\overline{\mathcal{A}}$ . In other words, a decomposable element is equal to a polynomial in elements of strictly lower degree. Therefore the highest degree of indecomposable invariants  $D_{\max} = D_{\max}(n, d)$  is equal to the least upper bound for the degrees of elements of a m.h.s.g. for  $R^{O(n)}$ . So using  $D_{\max}$  we can easily construct a finite system of generators. In this paper we give the following estimations on  $D_{\max}$  in case  $n = 3$ .

**Theorem 1.1.** *Let  $n = 3$  and  $d \geq 1$ . Then*

- If  $\text{char } \mathbb{F} = 3$ , then  $2d + 4 \leq D_{\max} \leq 2d + 7$ .
- If  $\text{char } \mathbb{F} \neq 2, 3$ , then  $D_{\max} = 6$ .

A m.h.s.g. for matrix  $GL(2)$ -invariants was found [16], [14], and [3] and for matrix  $O(2)$ -invariants over  $\mathbb{F} = \mathbb{C}$  in [16]. A m.h.s.g. for matrix  $GL(3)$ -invariants was established in [5] and [6]. For  $d = 2$  relations for matrix  $GL(3)$ -invariants were explicitly described in [12], [2]. In [10]  $D_{\max}$  was estimated for invariants of quivers of dimension  $(2, \dots, 2)$ .

The paper is organized as follows. In Section 2 we introduce notations that are used throughout the paper. We also formulate key Theorem 2.1 from [11].

In Section 3 we define an associative algebra  $A_{3,d}$  and find out some relations for it.

In Section 4 we consider relations for  $\overline{R^{O(n)}}$  of multidegree  $(1, \delta_2, \dots, \delta_d)$  for an arbitrary  $n$ . In Lemma 4.1 it is shown that the T-ideal of the mentioned relations is generated by two identities, whereas in Theorem 2.1 we take infinitely many identities.

In Section 5 we establish that T-ideal of relations for  $\overline{R^{O(3)}}$  is generated by four identities (see Theorem 5.1 together with the definition of  $A_{3,d}$ ). A connection between relations for  $R^{O(3)}$  and  $A_{3,d}$  is described in Corollary 5.5.

In Section 6 we apply results from Sections 3 and 5 to calculate  $D_{\max}$  and the nilpotency degree of  $A_{3,d}$  with deviation 3. Note that considering relations from Theorem 5.1 of degree less or equal than  $D_{\max}$  we obtain a *finite* generating system for the ideal of relations for  $\overline{R^{O(3)}}$ .

## 2 Notations and known results

For a vector  $\underline{t} = (t_1, \dots, t_u) \in \mathbb{N}^u$  we write  $\#\underline{t} = u$  and  $|\underline{t}| = t_1 + \dots + t_u$ . In this paper we use the following notions from [11]:

- the monoid  $\mathcal{M}$  (without unity) freely generated by *letters*  $x_1, \dots, x_d, x_1^T, \dots, x_d^T$ , the vector space  $\mathcal{M}_{\mathbb{F}}$  with the basis  $\mathcal{M}$ , and  $\mathcal{N} \subset \mathcal{M}$  the subset of primitive elements, where the notion of a primitive element is defined as above;
- the involution  $^T : \mathcal{M}_{\mathbb{F}} \rightarrow \mathcal{M}_{\mathbb{F}}$  defined by  $x^{TT} = x$  for a letter  $x$  and  $(a_1 \cdots a_p)^T = a_p^T \cdots a_1^T$  for  $a_1, \dots, a_p \in \mathcal{M}$ ;

- the equivalence  $y_1 \cdots y_p \sim z_1 \cdots z_p$  that holds if there exists a cyclic permutation  $\pi \in S_p$  such that  $y_{\pi(1)} \cdots y_{\pi(p)} = z_1 \cdots z_p$  or  $y_{\pi(1)} \cdots y_{\pi(p)} = z_p^T \cdots z_1^T$ , where  $y_1, \dots, y_p, z_1, \dots, z_p$  are letters;
- $\mathcal{M}_\sigma$ , a ring with unity of (commutative) polynomials over  $\mathbb{F}$  freely generated by “symbolic” elements  $\sigma_t(\alpha)$ , where  $t > 0$  and  $\alpha \in \mathcal{M}_\mathbb{F}$ ;
- $\mathcal{N}_\sigma$ , a ring with unity of (commutative) polynomials over  $\mathbb{F}$  freely generated by “symbolic” elements  $\sigma_t(\alpha)$ , where  $t > 0$  and  $\alpha \in \mathcal{N}$  ranges over  $\sim$ -equivalence classes; note that  $\mathcal{N}_\sigma \simeq \mathcal{M}_\sigma/L$ , where the ideal  $L$  is described in Lemma 3.1 of [11];
- $\mathbb{N}$ -gradings by degrees and  $\mathbb{N}^d$ -gradings by multidegrees for  $\mathcal{M}$ ,  $\mathcal{M}_\mathbb{F}$ ,  $\mathcal{N}$ , and  $\mathcal{N}_\sigma$ , where  $\text{mdeg}(a) = (\deg_{x_1} a + \deg_{x_1^T} a, \dots, \deg_{x_d} a + \deg_{x_d^T} a)$  for  $a \in \mathcal{M}$ ;
- the surjective homomorphism  $\Psi_n : \mathcal{N}_\sigma \rightarrow R^{O(n)}$ , defined in the natural way (see Section 1 of [11]);
- $\sigma_{t,r}(a, b, c) \in \mathcal{N}_\sigma$  introduced in [19], where  $a, b, c \in \mathcal{M}_\mathbb{F}$  and  $t, r \in \mathbb{N}$  (see Section 3 of [11] or see Definition 2.3 for the image of  $\sigma_{t,r}(a, b, c)$  in  $\overline{\mathcal{N}_\sigma}$ );
- a partial linearization  $\sigma_{\underline{t}; \underline{r}; \underline{s}}(\underline{a}; \underline{b}; \underline{c}) \in \mathcal{N}_\sigma$  of  $\sigma_{t,r}(a, b, c)$  (see Section 5 of [11]), where  $\underline{t} = (t_1, \dots, t_u) \in \mathbb{N}^u$ ,  $\underline{r} = (r_1, \dots, r_v) \in \mathbb{N}^v$ ,  $\underline{s} = (s_1, \dots, s_w) \in \mathbb{N}^w$  satisfy  $|\underline{r}| = |\underline{s}|$ ,  $\underline{a} = (a_1, \dots, a_u)$ ,  $\underline{b} = (b_1, \dots, b_v)$ ,  $\underline{c} = (c_1, \dots, c_w)$ , and  $a_i, b_j, c_k$  belong to  $\mathcal{M}_\mathbb{F}$  for  $1 \leq i \leq u$ ,  $1 \leq j \leq v$ ,  $1 \leq k \leq w$ . A formula for computation of  $\sigma_{\underline{t}; \underline{r}; \underline{s}}(\underline{a}; \underline{b}; \underline{c})$  is given in Lemma 5.2 of [11].

The mapping  $\Psi_n$  induces the isomorphism of algebras

$$\overline{R^{O(n)}} \simeq \overline{\mathcal{N}_\sigma} / \overline{K_n}$$

for some ideal  $\overline{K_n} \triangleleft \overline{\mathcal{N}_\sigma}$ . Elements of  $\overline{K_n}$  are called *relations* for  $\overline{R^{O(n)}}$ . The ideal of relations was described in Corollary 7.6 of [11]:

**Theorem 2.1.** *The ideal of relations  $\overline{K_n}$  is generated by  $\sigma_{t,r}(a, b, c)$ , where  $t + 2r > n$  and  $a, b, c$  range over  $\mathcal{M}_\mathbb{F}$ .*

**Remark 2.2.** Since  $\mathbb{F}$  is infinite, we can reformulate Theorem 2.1 as follows: the ideal  $\overline{K_n}$  is generated by  $\mathbb{N}^d$ -homogeneous elements  $\sigma_{\underline{t}; \underline{r}; \underline{s}}(\underline{a}; \underline{b}; \underline{c})$ , where  $|\underline{t}| + 2|\underline{r}| > n$  and  $a_i, b_j, c_k$  range over  $\mathcal{M}$ .

Consider  $a = y_1 \cdots y_p$ , where  $y_1, \dots, y_s$  are letters. We say that a letter  $y_i$  is followed by  $y_j$  in  $a$  if  $j = i + 1$  or  $j = 1$ ,  $i = p$ . For the sake of completeness let us recall the definition of  $\sigma_{t,r}(a, b, c) \in \overline{\mathcal{N}_\sigma}$ .

**Definition 2.3** (of  $\sigma_{t,r}(a, b, c) \in \overline{\mathcal{N}_\sigma}$ ). We assume  $d \geq 3$ . Denote by  $\overline{I_{t,r}} = \{a_k\}$  the set of primitive pairwise different words in letters  $x_i, x_i^T$ ,  $i = 1, 2, 3$ , satisfying

- $j_{a_k} \text{mdeg}(a_k) = (t, r, r)$  for some  $j_{a_k}$ ;

- the letters  $x_1, x_3, x_3^T$  are followed by  $x_1, x_2, x_2^T$  in  $a_k$ ;
- the letters  $x_1^T, x_2, x_2^T$  are followed by  $x_1^T, x_3, x_3^T$  in  $a_k$ .

Then

$$\sigma_{t,r}(x_1, x_2, x_3) = \sum_{a \in \overline{I_{t,r}}} (-1)^{\xi_a} \sigma_{j_a}(a) \text{ in } \overline{\mathcal{N}_\sigma},$$

where  $\xi_a = t + j_a(\deg_{x_2} a + \deg_{x_3} a + 1)$ . We also set  $\sigma_{0,0}(x_1, x_2, x_3) = 1$ . For any  $d \geq 1$  and  $a_1, a_2, a_3 \in \mathcal{M}_{\mathbb{F}}$  we define  $\sigma_{t,r}(a_1, a_2, a_3) \in \overline{\mathcal{N}_\sigma}$  as the result of substitution  $x_i \rightarrow a_i$ ,  $x_i^T \rightarrow a_i^T$ ,  $i = 1, 2, 3$ , in  $\sigma_{t,r}(x_1, x_2, x_3) \in \overline{\mathcal{N}_\sigma}$ .

The decomposition formula from [7] implies that for  $n \times n$  matrices  $A_i$ ,  $i = 1, 2, 3$ , with  $n = t_0 + 2r$ ,  $t_0 \geq 0$  we have

$$\text{DP}_{r,r}(A_1 + \lambda E, A_2, A_3) = \sum_{t=0}^{t_0} \lambda^{t_0-t} \sigma_{t,r}(A_1, A_2, A_3), \quad (1)$$

where  $\text{DP}_{r,r}(A_1, A_2, A_3)$  stands for the determinant-pfaffian (see [8]) and  $\sigma_{t,r}(A_1, A_2, A_3)$  is defined as the result of substitution  $a_i \rightarrow A_i$ ,  $a_i^T \rightarrow A_i^T$  in  $\sigma_{t,r}(a_1, a_2, a_3)$ . Thus  $\text{DP}_{r,r}$  relates to  $\sigma_{t,r}$  in the same way as the determinant relates to  $\sigma_t$ .

If  $f, h \in \overline{\mathcal{N}_\sigma}$  are equal as elements of  $\overline{R^{O(n)}}$ , then we write  $f \equiv h$ . In particular,  $f \in \overline{\mathcal{N}_\sigma}$  is a relation for  $\overline{R^{O(n)}}$  if and only if  $f \equiv 0$ . We say that  $f \equiv h$  follows from  $f_1 \equiv h_1, \dots, f_s \equiv h_s$ , if  $f - h$  is a linear combination of  $f_1 - h_1, \dots, f_s - h_s$  in  $\overline{\mathcal{N}_\sigma}$ . We use the following convention:

$$\text{tr}(a) = \sigma_1(a)$$

for all  $a \in \mathcal{M}_{\mathbb{F}}$ . Note that  $\text{tr}$  is linear in  $\mathcal{N}_\sigma$ , i.e.,  $\text{tr}(\alpha a + \beta b) = \alpha \text{tr}(a) + \beta \text{tr}(b)$  in  $\mathcal{N}_\sigma$  for  $\alpha, \beta \in \mathbb{F}$  and  $a, b \in \mathcal{M}_{\mathbb{F}}$  (see Lemma 3.1 of [11]). Let us remark that

$$\sigma_{\underline{t}; \underline{r}; \underline{s}}(\underline{a}; \underline{b}; \underline{c}) = \sigma_{\underline{t}; \underline{s}; \underline{r}}(\underline{a}^T; \underline{c}; \underline{b}) \text{ in } \mathcal{N}_\sigma, \quad (2)$$

where  $\underline{a}^T = (a_1^T, \dots, a_u^T)$ .

### 3 The algebra $A_{3,d}$

Given  $a \in \mathcal{M}_{\mathbb{F}}$ , we denote  $\bar{a} = a - a^T$ . For an algebra  $\mathcal{A}$ , denote by  $\text{id}\{a_1, \dots, a_s\}$  the ideal generated by  $a_1, \dots, a_s \in \mathcal{A}$ . We denote

- $N_{3,d} = \mathcal{M}_{\mathbb{F}} / \text{id}\{a^3 \mid a \in \mathcal{M}_{\mathbb{F}}\}$ ;
- $T(a, b, c) = a\bar{b}\bar{c} + \bar{b}a^T\bar{c} + \bar{b}\bar{c}a = a\bar{b}\bar{c} + \bar{b}a\bar{c} + \bar{b}\bar{c}a - \bar{b}a\bar{c}$ , where  $a, b, c \in \mathcal{M}_{\mathbb{F}}$ ;
- $A_{3,d} = N_{3,d} / \text{id}\{T(a, b, c) \mid a, b, c \in \mathcal{M}_{\mathbb{F}}\}$ ;
- $\mathcal{M}_1 = \mathcal{M} \sqcup \{1\}$ , i.e., we endow  $\mathcal{M}$  with the unity.

Since  $\mathbb{F}$  is infinite, the elements

- $T_1(a) = a^3$ ,
- $T_2(a, b) = a^2b + aba + ba^2$ ,
- $T_3(a, b, c) = abc + acb + bac + bca + cab + cba$ ,

where  $a, b, c \in \mathcal{M}_{\mathbb{F}}$ , are equal to zero in  $N_{3,d}$ . Moreover,

$$N_{3,d} = \mathcal{M}_{\mathbb{F}} / \text{id}\{T_1(a), T_2(a, b), T_3(a, b, c) \mid a, b, c \in \mathcal{M}\}.$$

In what follows we apply the following remark without references to it.

**Remark 3.1.** Consider  $f = \sum_i \alpha_i a_i b c_i$ , where  $\alpha_i \in \mathbb{F}$ ,  $a_i, c_i \in \mathcal{M}_1$ , and  $b \in \mathcal{M}$ . If  $f = 0$  in  $A_{3,d}$  is valid for all  $b \in \mathcal{M}$ , then  $f = 0$  in  $A_{3,d}$  is also valid for all  $b \in \mathcal{M}_{\mathbb{F}}$ .

The following equalities in  $\mathcal{M}$  are trivial:

$$\bar{a} = 2\bar{a}, \quad \overline{a^T} = -\bar{a} = \bar{a}^T, \quad \overline{ab} = ab - ba + \bar{b}a + b\bar{a} - \bar{b}\bar{a}.$$

**Lemma 3.2.** *We have*

- a) *If  $a \in \mathcal{M}$  and  $x$  is a letter, then  $a \in A_{3,d}$  is equal to the sum of the following elements:  $b_1, b_1xb_2, b_1x^2b_2, b_1x^2cxb_2$ , where  $b_1, b_2 \in \mathcal{M}_1$ ,  $\deg_x(b_i) = \deg_{x^T}(b_i) = 0$  for  $i = 1, 2$ , and  $c \in \mathcal{M}$ ; in particular, if  $\deg_x a > 3$ , then  $a = 0$  in  $A_{3,d}$ ;*
- b) *If  $\text{char } \mathbb{F} = 0$  or  $\text{char } \mathbb{F} > 3$ , then  $x_1 \cdots x_6 = 0$  in  $A_{3,d}$ ;*
- c)  *$(ab)^2 = b^2a^2$  in  $A_{3,d}$ , where  $a, b \in \mathcal{M}$ ; moreover,  $(a_1 \cdots a_s)^2 = a_s^2 \cdots a_1^2$  in  $A_{3,d}$ , where  $a_1, \dots, a_s \in \mathcal{M}$ ;*
- d)  *$ab \cdot c \cdot ba = -ca^2b^2 - b^2a^2c$  in  $A_{3,d}$ , where  $a, b, c \in \mathcal{M}$ ;*
- e) *If  $\text{char } \mathbb{F} = 3$  and for  $a \in \mathcal{M}$  we have  $\deg_{x_1}(a) = \deg_{x_2}(a) = 3$ , then  $a = \sum_i \alpha_i a_i x_1^2 x_2^2 x_1 x_2 b_i$  in  $A_{3,d}$ , where  $\alpha_i \in \mathbb{F}$  and  $a_i, b_i \in \mathcal{M}_1$ ;*
- f)  *$\bar{a}\bar{b}\bar{c} = -\bar{c}\bar{b}\bar{a}$  in  $A_{3,d}$ , where  $a, b, c \in \mathcal{M}$ .*

*Proof.* **a)** Let  $c \in \mathcal{M}$ . Considering  $T_2(x, c)$  we obtain

$$xcx = -x^2c - cx^2 \text{ in } A_{3,d}, \tag{3}$$

and considering  $T_2(x, xc)$  we obtain

$$xcx^2 = -x^2cx \text{ in } A_{3,d}. \tag{4}$$

The required follows from (3) and (4).

**b)** See Proposition 1 of [5].

c) The first part follows from part a) of Lemma 3.2 and the second part is a consequence of the first part.

d) Applying part a) of Lemma 3.2, we obtain  $ab \cdot c \cdot ba = a^2b^2c + cb^2a^2 + a^2cb^2 + b^2ca^2$  in  $A_{3,d}$ . To complete the proof, we consider  $T_3(a^2, b^2, c)$ .

e) See Statement 7 of [5].

f) The equality  $T(b, a, c) + T(b, c, a) - T_3(b, \bar{a}, \bar{c}) = 0$  in  $A_{3,d}$  gives the required.  $\square$

**Lemma 3.3.** *Let  $\text{char } \mathbb{F} = 3$  and  $1 \leq i \leq d$ . For any homogeneous  $e \in \mathcal{M}_{\mathbb{F}}$  of multidegree  $(\delta_1, \dots, \delta_d)$  with  $\delta_i < 3$  and  $\delta_1 + \dots + \delta_{i-1} + \delta_{i+1} + \dots + \delta_d > 0$  we define  $\pi_i(e) \in \mathcal{M}_{\mathbb{F}}$  as the result of substitution  $x_i \rightarrow 1$ ,  $x_i^T \rightarrow 1$  in  $a$ , where 1 stands for the unity of  $\mathcal{M}_1$ .*

*Then  $e = 0$  in  $A_{3,d}$  implies  $\pi_i(e) = 0$  in  $A_{3,d}$ .*

*Proof.* Let  $a, b, c \in \mathcal{M}_{\mathbb{F}}$ . By definition,  $\pi_i(ab) = \pi_i(a)\pi_i(b)$ . It is not difficult to see that  $\pi_i(\bar{a}) = \pi_i(a)$ . Then by straightforward calculations we can show that  $\pi_i(T_2(a, b)) = 0$ ,  $\pi_i(T_3(a, b, c)) = 0$ , and  $\pi_i(T(a, b, c)) = 0$  in  $A_{3,d}$ . The proof is completed.  $\square$

**Lemma 3.4.** *The equality*

$$\bar{a}ub\bar{v}\bar{c}w\bar{e} = 0$$

*holds in  $A_{3,d}$  for all  $a, b, c, e \in \mathcal{M}$  and  $u, v, w \in \mathcal{M}_1$ .*

*Proof.* In this proof all elements belong to  $\mathcal{M}$  and all equalities are considered in  $A_{3,d}$ .

Consider  $T(\bar{a}\bar{b}, c, e) = 0$ . Using  $\overline{\bar{a}\bar{b}} = \bar{a}\bar{b} - \bar{b}\bar{a}$  and part f) of Lemma 3.2, we obtain

$$\bar{a}\bar{b}\bar{c}\bar{e} = 0. \quad (5)$$

Equalities  $T(u, a, b)\bar{c}\bar{e} = 0$  and  $\bar{a}\bar{b}T(u, c, e) = 0$  together with (5) imply

$$\bar{a}ub\bar{c}\bar{e} = -\bar{a}\bar{b}u\bar{c}\bar{e} = \bar{a}\bar{b}\bar{c}u\bar{e}. \quad (6)$$

Hence  $\bar{a}T(u, b, c)\bar{e} = 0$  implies

$$\bar{a}ub\bar{c}\bar{e} = \bar{a}\bar{b}u\bar{c}\bar{e} = \bar{a}\bar{b}\bar{c}u\bar{e} = 0. \quad (7)$$

It follows from  $T_3(\bar{a}u, \bar{b}\bar{c}, v\bar{e}) = 0$  together with (5) and (7) that

$$\bar{a}ub\bar{c}v\bar{e} = 0. \quad (8)$$

Thus  $\bar{a}uT_3(\bar{b}, v, \bar{c}\bar{e}) = 0$  and  $T_3(\bar{a}\bar{b}, u, \bar{c})v\bar{e} = 0$  imply

$$\bar{a}ub\bar{v}\bar{c}\bar{e} = \bar{a}\bar{b}u\bar{c}v\bar{e} = 0. \quad (9)$$

Considering  $T_3(\bar{a}u, \bar{b}\bar{v}\bar{c}, w\bar{e}) = 0$ , we obtain

$$\bar{a}ub\bar{v}\bar{c}w\bar{e} = 0. \quad (10)$$

The claim of the lemma is proved.  $\square$

**Lemma 3.5.** *Let  $\text{char } \mathbb{F} = 3$ . For  $0 \leq s \leq 4$  we assume that  $x_1, \dots, x_{8-2s}$  are pairwise different letters,  $a_1, \dots, a_s, v, w \in \mathcal{M}$ , and  $u_1, \dots, u_{s+1} \in \mathcal{M}_1$ . If  $\deg_{x_i}(u_1 \cdots u_{s+1}) = 3$  for  $1 \leq i \leq 8 - 2s$ , then the following equalities hold in  $A_{3,d}$ :*

a)  $u_1 \bar{a}_1 \cdots u_s \bar{a}_s u_{s+1} = 0$ ;

b) a product of  $vw - wv$ ,  $\bar{a}_1, \dots, \bar{a}_{s-1}$ ,  $u_1, \dots, u_{s+1}$  in any succession is equal to zero, where  $s > 0$ .

In particular, if  $u \in \mathcal{M}_1$  and  $\deg_{x_i}(u) = 3$  for  $1 \leq i \leq 8$ , then  $u = 0$ .

*Proof.* In this proof all equalities are considered in  $A_{3,d}$ . We prove by decreasing induction on  $s$ .

1. Let  $s = 4$ . Then part a) follows from Lemma 3.4.

2. Assume that part a) holds for some  $s > 0$ . We set  $a_s = vw$  in part a). Thus,

$$u_1 \bar{a}_1 \cdots u_{s-1} \bar{a}_{s-1} u_s (vw - wv) u_{s+1} = 0.$$

Considering  $a_i$  with  $1 \leq i < s$  instead of  $a_s$ , we complete the proof of part b) for the given  $s$ .

3. Assume that part b) holds for some  $s = k + 1 > 0$ . We claim that part a) is valid for  $s = k$ . We set  $W = x_1^2 x_2^2 x_1 x_2$ . By part e) of Lemma 3.2,  $u_1 \bar{a}_1 \cdots u_k \bar{a}_k u_{k+1}$  is a linear combination of elements

$$v_1 w_1 \cdots v_{k+1} w_{k+1} v_{k+2}, \quad (11)$$

where pairwise different  $w_1, \dots, w_{k+1}$  belong to the set  $\{\bar{a}_1, \dots, \bar{a}_k, W\}$  and  $v_1, \dots, v_{k+2}$  range over  $\mathcal{M}_1$ . For  $v = x_1^2 x_2^2$  and  $w = x_1$  we have  $W = vw x_2$  and  $wv x_2 = 0$ . Part b) for  $s = k + 1$  implies that all elements from (11) are zero and the claim is proved.  $\square$

## 4 Relations for $R^{O(n)}$ of multidegree $(1, \delta_2, \dots, \delta_d)$

In this section we assume that  $n$  is arbitrary.

**Lemma 4.1.** Any homogeneous relation of  $\overline{R^{O(n)}}$  of multidegree  $(1, \delta_2, \dots, \delta_d)$  follows from relations:

- $\sigma_{1,t-1;r;r}(e, a; b; c) \equiv 0$ ,
- $\sigma_{t;1,r-1;r}(a; e, b; c) \equiv 0$ ,

where  $t + 2r = n + 1$ ,  $a, b, c$  range over  $\mathcal{M}_{\mathbb{F}}$ , and  $e$  ranges over elements of  $\mathcal{M}$  such that  $\deg_{x_1}(e) + \deg_{x_1^T}(e) = 1$ .

*Proof.* We assume that  $f = \sigma_{\underline{t};\underline{r};\underline{s}}(\underline{a}; \underline{b}; \underline{c})$  is a homogeneous relation of multidegree  $(1, \delta_2, \dots, \delta_d)$ , where  $|\underline{t}| + 2|\underline{r}| > n$ ,  $a_i, b_j, c_k \in \mathcal{M}$ , and consider the set of all  $t_i, r_j, s_k$ . Then at least one element of this set is equal to 1. We set  $u = \# \underline{a}$ ,  $v = \# \underline{b}$ , and  $w = \# \underline{c}$ .

If  $t_1 = 1$ , then applying Lemma 5.2 of [11] it is not difficult to see that the next equality holds in  $\overline{\mathcal{N}_\sigma}$ :

$$\begin{aligned} f = & - \sum_{i=2}^u \sigma_{\Delta_i}(a_1 a_i, a_2, \dots, a_u; \underline{b}; \underline{c}) - \sum_{j=1}^v \sigma_{\Theta_j}(a_2, \dots, a_u; a_1 b_j, b_1, \dots, b_v; \underline{c}) + \\ & + \sum_{j=1}^v \sigma_{\Theta_j}(a_2, \dots, a_u; a_1 b_j^T, b_1, \dots, b_v; \underline{c}), \end{aligned}$$

where  $\Delta_i = (1, t_2, \dots, t_i - 1, \dots, t_u; \underline{r}; \underline{s})$  and  $\Theta_j = (t_2, \dots, t_u; 1, r_1, \dots, r_j - 1, \dots, r_v; \underline{s})$ .

If  $r_1 = 1$ , then the next equality holds in  $\overline{\mathcal{N}}_\sigma$ :

$$f = - \sum_{i=1}^u \sigma_{\Delta_i}(\underline{a}; b_1 a_i^T, b_2, \dots, b_v; \underline{c}) - \sum_{k=1}^w \sigma_{\Theta_k}(b_1 c_k, a_1, \dots, a_u; b_2, \dots, b_v; \underline{c}) + \\ + \sum_{k=1}^w \sigma_{\Theta_k}(b_1 c_k^T, a_1, \dots, a_u; b_2, \dots, b_v; \underline{c}),$$

where  $\Delta_i = (t_1, \dots, t_i - 1, \dots, t_u; 1, r_2, \dots, r_v; \underline{s})$ ,  $\Theta_k = (1, t_1, \dots, t_u; r_2, \dots, r_v; s_1, \dots, s_k - 1, \dots, s_w)$ .

All other cases can be treated similar. We repeat this procedure and use (2) to obtain the required.  $\square$

## 5 Reduction to $A_{3,d}$

In this section we assume  $n = 3$ . This section is dedicated to the proof of the following theorem and its corollary. We say that  $f \in \mathcal{M}_{\mathbb{F}}$  does not contain a letter  $x$  if  $f$  is a linear combination of  $a_1, \dots, a_s \in \mathcal{M}$ , where  $\deg_x a_i = 0$  for all  $i$ .

**Theorem 5.1.** *Any relation for  $\overline{R^{O(3)}}$  follows from relations:*

- (A)  $\text{tr}(au) \equiv 0$  for  $u \in \mathcal{M}_{\mathbb{F}}$  such that  $u = 0$  in  $A_{3,d}$ ;
- (B)  $\sigma_3(ab) \equiv 0$ ;
- (C)  $\sigma_t(a) \equiv 0$  for  $t > 3$ ;

where  $a, b \in \mathcal{M}$ .

**Lemma 5.2.**

- a)  $\sigma_2(a) = -\frac{1}{2} \text{tr}(a^2)$  and  $3\sigma_3(a) = \text{tr}(a^3)$  in  $\overline{\mathcal{N}}_\sigma$ , where  $a \in \mathcal{M}_{\mathbb{F}}$ ;
- b) A relation  $h \equiv 0$ , where  $h$  has multidegree  $(1, \delta_2, \dots, \delta_d)$ , follows from (A).
- c) A relation  $\sigma_{t,r}(a, b, c) \equiv 0$ , where  $t + 2r > 3$ ,  $a, b, c \in \mathcal{M}_{\mathbb{F}}$ , and  $t$  or  $r$  is not divisible by  $\text{char } \mathbb{F}$  (or  $\text{char } \mathbb{F} = 0$ ), follows from (A).
- d) If  $a \in \mathcal{M}$  and  $\deg_{x_1} a > 3$ , then  $\text{tr}(a) \equiv 0$  follows from (A);
- e) If  $\text{char } \mathbb{F} = 3$  and  $u = 0$  in  $A_{3,d}$ , then  $\text{tr}(u) \equiv 0$  follows from (A);

*Proof.* **a)** Since  $\text{tr}(a^2) = \text{tr}(a)^2 - 2\sigma_2(a)$  in  $\mathcal{N}_\sigma$  (see Lemma 3.1 from [11]), we obtain the first equality. The second equality is can be proved similarly.

**b)** By Lemma 4.1, it is enough to show that  $f_1 = \sigma_{1,t-1;r,r}(e, a; b; c) \equiv 0$  and  $f_2 = \sigma_{t,1,r-1;r}(a; e, b; c) \equiv 0$  with  $t + 2r = 4$  follow from (A), where  $a, b, c \in \mathcal{M}_{\mathbb{F}}$  and  $e \in \mathcal{M}$ . To complete the proof we consider the following cases.



If  $t = 2$  and  $r = 1$ , then  $f_1 = -\text{tr}(eT(a, b, c))$  and  $f_2 = -\text{tr}(eT_2(a, \bar{c})) + \text{tr}(eT(a, a, c))$  in  $\overline{\mathcal{N}_\sigma}$ .  
 If  $t = 4$  and  $r = 0$ , then  $f_1 = -\text{tr}(ea^3)$  in  $\overline{\mathcal{N}_\sigma}$ .  
 If  $t = 0$  and  $r = 2$ , then  $f_2 = -\text{tr}(e\bar{c}b\bar{c}) = -\text{tr}(eT(\bar{c}, b, c))$  in  $\overline{\mathcal{N}_\sigma}$ .

c) If  $t$  is not divisible by  $\text{char } \mathbb{F}$ , then  $\sigma_{t,r}(a, b, c) = \frac{1}{t}\sigma_{t-1,1;r;r}(a, a; b; c)$  (see Lemma 5.5 of [11]). Part b) concludes the proof. The other case is alike.

d) It follows from the proof of part a) of Lemma 3.2 and linearity of  $\text{tr}$ .

e) For  $a, b, c \in \mathcal{M}$  we have  $\text{tr}(T_2(a, b)) = 3\text{tr}(a^2b)$ ,  $\text{tr}(T_3(a, b, c)) = 3\text{tr}(abc) + 3\text{tr}(acb)$ , and  $\text{tr}(T(a, b, c)) = 3\text{tr}(a\bar{b}\bar{c})$  in  $\mathcal{N}_\sigma$ . By part a) of Lemma 5.2, we have  $\text{tr}(T_1(a)) = 3\sigma_3(a)$  in  $\overline{\mathcal{N}_\sigma}$ . The proof is completed.  $\square$

**Lemma 5.3.** *We have that (A) and (B) are relations for  $\overline{R^{O(3)}}$ .*

*Proof.* The proof of part b) of Lemma 5.2 implies that  $\text{tr}(ea^3) \equiv 0$  and  $\text{tr}(eT(a, b, c)) \equiv 0$  in  $\overline{R^{O(3)}}$  for all  $a, b, c \in \mathcal{M}_\mathbb{F}$  and  $e \in \mathcal{M}$ . The fact that  $\text{tr}$  is linear shows that (A) is a relation.

Since  $\sigma_3(AB) = \sigma_3(A)\sigma_3(B)$  for all  $3 \times 3$  matrices  $A$  and  $B$  over the polynomial ring  $R$  (see Section 1), we obtain  $\sigma_3(ab) \equiv 0$ .  $\square$

**Lemma 5.4.** *Relations  $\sigma_{0,3}(v, w) \equiv 0$  and  $\sigma_{3,3}(u, v, w) \equiv 0$ , where  $u, v, w \in \mathcal{M}_\mathbb{F}$ , follow from (A), (B), and (C).*

*Proof.* Clearly, without loss of generality we can assume that  $u = x_1$ ,  $v = x_2$ ,  $w = x_3$  are letters and  $d \geq 3$ . Consider  $t, r \geq 0$  such that  $t + 2r > 3$  and  $t$  or  $r$  is odd.

Let  $x \in \mathcal{M}$  be a letter and  $a = b_0a_1b_1 \cdots a_pb_p \in \mathcal{M}$ , where  $a_1, \dots, a_p \in \mathcal{M}$  are words in  $x, x^T$  and  $b_1, \dots, b_{p-1} \in \mathcal{M}$ ,  $b_0, b_p \in \mathcal{M}_1$  do not contain  $x, x^T$ , i.e.,  $\deg_x(b_i) = \deg_{x^T}(b_i) = 0$  for  $0 \leq i \leq p$ . We say that  $a$  is *fixed* by  $x$  if there is no non-trivial cyclic permutation  $\pi \in S_p$  such that

$$a_1 = a_{\pi(1)}, \dots, a_p = a_{\pi(p)} \text{ or } a_1^T = a_{\pi(p)}, \dots, a_p^T = a_{\pi(1)}.$$

Assume that  $a = b_0a_1b_1 \cdots a_pb_p$  is fixed by  $x$  and  $b_0 \cdots b_p = y_1 \cdots y_s$ , where  $y_i$  is a letter for all  $i$ . Then the following elements

$$c_0a_1c_1 \cdots a_pc_p,$$

where  $c_1, \dots, c_{p-1} \in \mathcal{M}$ ,  $c_0, c_p \in \mathcal{M}_1$ , and  $c_0 \cdots c_p = z_{\rho(1)} \cdots z_{\rho(s)}$  for  $z_i \in \{y_i, y_i^T\}$  ( $1 \leq i \leq s$ ) and  $\rho \in S_s$ , are primitive and pairwise different with respect to the  $\sim$ -equivalence.

For  $x \in \{x_1, x_2, x_3\}$  we denote  $Y = \{x_2, x_3\} \setminus \{x\}$ . The above mentioned property together with the definition of  $\sigma_{t,r}$  implies that

$$\sigma_{t,r}(x_1, x_2, x_3) = f_1 + f_2 + f_3 \text{ in } \overline{\mathcal{N}_\sigma}. \quad (12)$$

Here

- $f_1 = \sum (-1)^{\xi_a} \text{tr}(a)|_{y \rightarrow \bar{y}, y \in Y}$ , where the sum ranges over  $a \in \overline{I_{t,r}}$  such that  $a$  is fixed by  $x$  and  $\deg_y(a) = r$  for all  $y \in Y$ ;
- $f_2 = \sum (-1)^{\xi_a} \text{tr}(a)$ , where the sum ranges over  $a \in \overline{I_{t,r}}$  such that  $a$  is not fixed by  $x$ ;

- $f_3 \equiv 0$  follows from (B) and (C).

Let us consider  $\sigma_{t,r}$  from the formulation of the lemma.

- a) Let  $t = 0$  and  $r = 3$ . We take  $x = x_2$ . By part c) of Lemma 3.2,

$$f_1 = \text{tr}(x_2 \bar{x}_3 x_2 \bar{x}_3 x_2^T \bar{x}_3) \equiv \text{tr}(\bar{x}_3^2 x_2^2 x_2^T \bar{x}_3) \equiv 0 \text{ and}$$

$$f_2 = \text{tr}(x_2 x_3 x_2 x_3 x_2 x_3^T) - \text{tr}(x_2 x_3 x_2 x_3^T x_2 x_3^T) \equiv \text{tr}(x_3^2 x_2^2 x_2 x_3^T) - \text{tr}(x_2 x_3 (x_3^T)^2 x_2^2) \equiv 0$$

follow from (A).

b) Let  $t = r = 3$ . We take  $x = x_1$ . By Lemma 3.4,  $f_1 \equiv 0$  follows from (A). Clearly,  $a \in \overline{I_{t,r}}$  is not fixed by  $x$  if and only if  $a \sim x a_1 x a_2 x a_3$ , where  $a_1, a_2, a_3 \in \mathcal{M}$ . Thus,  $f_2 = h_1 + h_2$  in  $\overline{\mathcal{N}_\sigma}$ , where

$$\begin{aligned} h_1 = & \text{tr}(x_1 x_2^T x_3^T x_1 x_2^T x_3 x_1 x_2 x_3^T) + \text{tr}(x_1 x_2^T x_3^T x_1 x_2 x_3^T x_1 x_2^T x_3) \\ & - \text{tr}(x_1 x_2^T x_3^T x_1 x_2^T x_3 x_1 x_2 x_3) - \text{tr}(x_1 x_2^T x_3^T x_1 x_2 x_3^T x_1 x_2 x_3) \\ & - \text{tr}(x_1 x_2^T x_3^T x_1 x_2 x_3 x_1 x_2^T x_3) - \text{tr}(x_1 x_2^T x_3^T x_1 x_2 x_3 x_1 x_2 x_3^T) \\ & + \text{tr}(x_1 x_2^T x_3 x_1 x_2 x_3^T x_1 x_2 x_3) + \text{tr}(x_1 x_2^T x_3 x_1 x_2 x_3 x_1 x_2 x_3^T) \end{aligned}$$

and  $h_2$  is a linear combination of elements  $h = \text{tr}(x_1 y_2 y_3 x_1 y_2 y_3 x_1 z_2 z_3)$ , where  $y_i, z_i \in \{x_i, x_i^T\}$  for  $i = 2, 3$ . Since  $x_1 y_2 y_3 \cdot x_1 y_2 y_3 \cdot x_1 = y_3^2 y_2^2 x_1^2 \cdot x_1 = 0$  in  $A_{3,d}$  (see part c) of Lemma 3.2),  $h \equiv 0$  follows from (A).

Using the equality  $b^T = b - \bar{b}$ , we can get rid of  $x_2^T$  and  $x_3^T$  in  $h_1$ . Hence

$$\begin{aligned} h_1 = & \text{tr}(x_1 \bar{x}_2 \bar{x}_3 x_1 \bar{x}_2 x_3 x_1 x_2 \bar{x}_3) + \text{tr}(x_1 \bar{x}_2 \bar{x}_3 x_1 x_2 \bar{x}_3 x_1 \bar{x}_2 x_3) \\ & + \text{tr}(x_1 \bar{x}_2 x_3 x_1 x_2 \bar{x}_3 x_1 x_2 x_3) + \text{tr}(x_1 \bar{x}_2 x_3 x_1 x_2 x_3 x_1 x_2 \bar{x}_3) + h_3 \end{aligned}$$

in  $\overline{\mathcal{N}_\sigma}$ , where  $h_3$  is a linear combination of elements  $h$  as above. By Lemma 3.4,

$$\text{tr}(x_1 \bar{x}_2 \bar{x}_3 x_1 \bar{x}_2 x_3 x_1 x_2 \bar{x}_3) + \text{tr}(x_1 \bar{x}_2 \bar{x}_3 x_1 x_2 \bar{x}_3 x_1 \bar{x}_2 x_3) \equiv 0$$

follows from (A). Applying part d) of Lemma 3.2 to  $x_3 \cdot x_1 x_2 \cdot \bar{x}_3 \cdot x_1 x_2 \cdot x_3$  and using part c) of Lemma 3.2, we obtain

$$\text{tr}(x_1 \bar{x}_2 x_3 x_1 x_2 \bar{x}_3 x_1 x_2 x_3) + \text{tr}(x_1 \bar{x}_2 x_3 x_1 x_2 x_3 x_1 x_2 \bar{x}_3) \equiv -\text{tr}(x_1 \bar{x}_2 \bar{x}_3 x_3^2 x_2^2 x_1^2) \equiv 0$$

follows from (A). The proof is completed.  $\square$

*Proof of Theorem 5.1.* By Theorem 2.1 and Lemma 5.3, it is enough to show that  $f = \sigma_{t,r}(a, b, c) \equiv 0$ , where  $t + 2r > 3$  and  $a, b, c \in \mathcal{M}_{\mathbb{F}}$ , follows from (A), (B), and (C).

Applying part a) of Lemma 5.2, we obtain that  $f = f_1 + f_2 + f_3$  in  $\overline{\mathcal{N}_\sigma}$ , where  $f_1 = \sum_i \alpha_i \text{tr}(a_i)$ ,  $f_2 = \sum_j \beta_j \sigma_3(b_j)$ ,  $\alpha_i, \beta_j \in \mathbb{F}$ ,  $a_i, b_j \in \mathcal{M}$ , and  $f_3$  follows from (C). Since  $\deg f > 3$ ,  $f_2 \equiv 0$  follows from (B).

If  $t > 6$  or  $r > 6$ , then  $f_1 \equiv 0$  follows from (A). To prove this claim, note that if  $t > 6$ , then  $\deg_a a_i > 3$  or  $\deg_{a^T} a_i > 3$ , where we consider  $a_i$  as a word in  $a, b, c, a^T, b^T, c^T$ . Part d) of Lemma 5.2 implies that  $\text{tr}(a_i) \equiv 0$  follows from (A).

a) Let  $\text{char } \mathbb{F} = 0$  or  $\text{char } \mathbb{F} \geq 5$ . Part b) of Lemma 3.2 implies that if  $\deg f > 6$ , then  $f_1 \equiv 0$  follows from (A). If  $\deg f \leq 6$ , then  $f \equiv 0$  follows from (A) by part c) of Lemma 5.2 or  $f = \sigma_5(x) \equiv 0$  follows from (C), where  $x$  is a letter.

**b)** Let  $\text{char } \mathbb{F} = 3$ . In the rest of the proof we apply part e) of Lemma 5.2 without reference to it. We have  $f_1 = \sum_k \gamma_k \text{tr}(c_k)$ , where  $\gamma_k \in \mathbb{F}$  and  $c_k$  is a monomial in  $a, b, c, \bar{a}, \bar{b}, \bar{c}$  for all  $k$ .

We assume  $r = 6$ . If  $\deg_b c_k = \deg_c c_k = \deg_{\bar{b}} c_k = \deg_{\bar{c}} c_k = 3$ , then  $\text{tr}(c_k) \equiv 0$  follows from (A) by Lemma 3.4; otherwise,  $\text{tr}(c_k) \equiv 0$  follows from (A) by part d) of Lemma 5.2.

We claim that if  $t = 6$  and  $r = 3$ , then  $\text{tr}(c_k) \equiv 0$  follows from (A) for all  $k$ . Assume that this claim does not hold. Then  $\deg_a(c_k) = \deg_{\bar{a}}(c_k) = 3$  by part d) of Lemma 5.2 and  $\deg_b(c_k) = \deg_c(c_k) = 3$  by Lemma 3.4. Part e) of Lemma 3.2 implies that  $c_k = \alpha W_{a\bar{a}}W_{bc} + \beta W_{bc}W_{a\bar{a}}$  in  $A_{3,d}$ , where  $\alpha, \beta \in \mathbb{F}$  and  $W_{uv} = u^2v^2uv$  for any  $u, v \in \mathcal{M}_{\mathbb{F}}$ . We set  $v = b^2c^2$ ,  $w = b$ , and  $s = 4$  in part b) of Lemma 3.5 and obtain  $W_{a\bar{a}}W_{bc} = W_{bc}W_{a\bar{a}} = 0$  in  $A_{3,d}$ . Hence  $\text{tr}(c_k) \equiv 0$  follows from (A); a contradiction. Thus the claim is proved.

If  $t = 6$  and  $r = 0$ , then it is not difficult to see that  $f = \sigma_6(a) \equiv 0$  follows from (A), (B), and (C) by Lemma 9 of [5].

Part c) of Lemma 5.2 and Lemma 5.4 complete the consideration of the case of  $\text{char } \mathbb{F} = 3$ .  $\square$

**Corollary 5.5.** *Let  $u, v \in \mathcal{M}_{\mathbb{F}}$  do not contain  $x_1, x_1^T$ . We have*

a)  $\text{tr}(ux_1) \equiv 0$  if and only if  $u = 0$  in  $A_{3,d}$ ;

b) if  $\text{char } \mathbb{F} = 3$ , then  $\text{tr}(ux_1^2) \equiv 0$  if and only if  $u = 0$  in  $A_{3,d}$ .

*Proof.* **a)** If  $u = 0$  in  $A_{3,d}$ , then  $\text{tr}(ux_1) \equiv 0$  by Lemma 5.3.

If  $\text{tr}(ux_1) \equiv 0$ , then  $\text{tr}(ux_1) = \sum_i \alpha_i \text{tr}(u_i a_i)$  in  $\overline{\mathcal{N}}_{\sigma}$ , where  $\alpha_i \in \mathbb{F}$ ,  $a_i \in \mathcal{M}$ ,  $u_i \in \mathcal{M}_{\mathbb{F}}$  is  $\mathbb{N}^d$ -homogeneous with  $u_i = 0$  in  $A_{3,d}$ , and  $\deg_{x_1}(a_i u_i) + \deg_{x_1^T}(a_i u_i) = 1$  (see Theorem 5.1). For  $a, b, c, e \in \mathcal{M}_{\mathbb{F}}$  the following equalities in  $\overline{\mathcal{N}}_{\sigma}$  hold:

$$\text{tr}(T_2(a, b) e) = \text{tr}(T_2(a, e) b),$$

$$\text{tr}(T_3(a, b, c) e) = \text{tr}(T_3(e, b, c) a),$$

$$\text{tr}(T(a, b, c) e) = \text{tr}(T(e, b, c) a),$$

$$\text{tr}(T(a, b, c) e) = \text{tr}((T_3(a, e, \bar{c}) - T(a, e, c) - T(e, a, c)) b),$$

$$\text{tr}(T(a, b, c) e) = \text{tr}((T_3(a, \bar{b}, e) - T(a, b, e) - T(e, b, a)) c).$$

Thus  $\text{tr}(ux_1) = \sum_j \beta_j \text{tr}(v_j x_1)$  in  $\overline{\mathcal{N}}_{\sigma}$ , where  $\beta_j \in \mathbb{F}$  and  $v_j \in \mathcal{M}_{\mathbb{F}}$  satisfies  $v_j = 0$  in  $A_{3,d}$ . Therefore  $u = \sum_j \beta_j v_j$  in  $\mathcal{M}_{\mathbb{F}}$  and the proof is completed.

**b)** If  $\text{tr}(ux_1^2) \equiv 0$ , then  $\text{tr}(u(x_1 + x_{d+1})^2) \equiv 0$ , where  $x_{d+1}$  stands for a new letter. Taking homogeneous components of multidegrees  $(1, \delta_2, \dots, \delta_d, 1)$  we obtain  $\text{tr}((ax_1 + x_1 a)x_{d+1}) \equiv 0$ . By part a),  $ux_1 + x_1 u = 0$  in  $A_{3,d}$ . Application of  $\pi_1$  from Lemma 3.3 completes the proof.  $\square$

## 6 $D_{max}$ and the nilpotency degree of $A_{3,d}$

Theorem 1.1 is a consequence of Lemma 6.3 which is proved in this section. For an  $n \times n$  matrix  $X$  we denote  $\bar{X} = X - X^T$ .

**Lemma 6.1.** *The invariant  $\text{tr}(X_1^2 \bar{X}_1^2 X_1 \bar{X}_1) \in R^{O(3)}$  is indecomposable, i.e.,  $\text{tr}(x_1^2 \bar{x}_1^2 x_1 \bar{x}_1) \neq 0$ .*

*Proof.* For short, we write  $X$  instead of  $X_1$ . Assume that the claim of this lemma does not hold. By part d) of Lemma 5.2, any indecomposable element of  $R^{O(3)}$  of multidegree  $(k)$ , where  $k < 6$ , is a linear combination of the following elements:

$$\text{tr}(X), \sigma_2(X), \sigma_3(X), \text{tr}(XX^T), \text{tr}(X^2 X^T), \text{tr}(X^2 (X^T)^2), \sigma_2(XX^T). \quad (13)$$

Then

$$\text{tr}(X^2 \bar{X}^2 X \bar{X}) = \sum_i \alpha_i f_{i_1} \cdots f_{i_q}, \quad (14)$$

where  $\alpha_i \in \mathbb{F}$  and  $f_j$  is an element from (13). Moreover, if we take an arbitrary  $3 \times 3$  matrix over  $R$  instead of  $X$ , then (13) remains valid and coefficients  $\alpha_i$  do not depend on  $X$ . Denote by  $\alpha, \beta, \gamma \in \mathbb{F}$ , respectively, the coefficients of  $\text{tr}(XX^T)^3$ ,  $\text{tr}(XX^T) \text{tr}(X^2 (X^T)^2)$ , and  $\text{tr}(XX^T) \sigma_2(XX^T)$ , respectively, in (14). For  $a, b \in R$  we set

$$X = \begin{pmatrix} 0 & a & 0 \\ 0 & 0 & b \\ 0 & 0 & 0 \end{pmatrix}.$$

If we take  $b = 0$ , then  $\alpha = 0$ . If we assume  $a, b \neq 0$ , then

$$\alpha \text{tr}(XX^T)^3 + \beta \text{tr}(XX^T) \text{tr}(X^2 (X^T)^2) + \gamma \text{tr}(XX^T) \sigma_2(XX^T) = 0.$$

Thus

$$a^4 b^2 (1 + \beta + \gamma) + a^2 b^4 (\beta + \gamma) = 0$$

for all  $a, b \in R$ ; a contradiction.  $\square$

**Lemma 6.2.** *Let  $\text{char } \mathbb{F} = 3$ . Then*

- a) *for any  $a \in \mathcal{M}$  satisfying  $a \neq 0$  in  $A_{3,d}$  we have  $\deg a \leq 2d + 7$ .*
- b) *there exists an  $a \in \mathcal{M}$  such that  $a \neq 0$  in  $A_{3,d}$  and  $\deg a = 2d + 4$ .*

*Proof.* **a)** In this proof all equalities are considered in  $A_{3,d}$ . We can assume that  $a$  is a word in  $x_1, \dots, x_d, \bar{x}_1, \dots, \bar{x}_d$ . By part a) of Lemma 3.2,  $\deg_{x_i}(a) \leq 3$  for all  $i$ . Let  $k = \deg_{\bar{x}_1}(a) + \dots + \deg_{\bar{x}_d}(a)$ . By Lemma 3.4,  $k < 4$ . Moreover, by part a) of Lemma 3.5 there are no pairwise different  $1 \leq i_1, \dots, i_{8-2k} \leq d$  such that  $\deg_{x_{i_j}}(a) = 3$  for all  $1 \leq j \leq 8 - 2k$ . Thus,  $\deg a \leq k + 2d + (7 - 2k) \leq 2d + 7$ .

**b)** Consider  $a_d = x_1^2 \bar{x}_1^2 x_1 \bar{x}_1 x_2^2 \cdots x_d^2 \in \mathcal{M}$ . If  $d = 1$ , then  $a_d \neq 0$  in  $A_{3,d}$  by part e) of Lemma 5.2 and Lemma 6.1. We assume that  $d > 1$ . If  $a_d = 0$  in  $A_{3,d}$ , then applying  $\pi_d, \dots, \pi_2$  from Lemma 3.3 to  $a_d$  we obtain that  $a_1 = 0$  in  $A_{3,d}$ ; a contradiction.  $\square$

**Lemma 6.3.**

a) If  $\text{char } \mathbb{F} = 3$  and  $d \geq 1$ , then  $2d + 4 \leq D_{\max} \leq 2d + 7$ .

b) If  $\text{char } \mathbb{F} \neq 2$  and  $d = 1$ , then  $D_{\max} = 6$ .

c) If  $\text{char } \mathbb{F} \neq 2, 3$  and  $d \geq 1$ , then  $D_{\max} = 6$ .

*Proof.* **a)** Let  $\text{tr}(a) \neq 0$ , where  $a \in \mathcal{M}_{\mathbb{F}}$ . By part e) of Lemma 5.2,  $a \neq 0$  in  $A_{3,d}$ . Part a) of Lemma 6.2 implies  $\deg a \leq 2d + 7$ . Thus part a) of Lemma 5.2 and equality (B) from Theorem 5.1 show that  $D_{\max} \leq 2d + 7$ .

Consider  $a_d = x_1^2 \bar{x}_1^2 x_1 \bar{x}_1 x_2^2 \cdots x_d^2 \in \mathcal{M}$ . If  $d = 1$ , then  $\text{tr}(a_d) \neq 0$  by Lemma 6.1. We assume that  $d > 1$ . If  $\text{tr}(a_d) = 0$ , then  $a_{d-1} = 0$  in  $A_{3,d}$  by part b) of Corollary 5.5; a contradiction with the proof of part b) of Lemma 6.2.

**b)** Let  $a \in \mathcal{M}_{\mathbb{F}}$  and  $d = 1$ . If  $\deg a > 6$ , then  $\deg_{x_1}(a) > 3$  or  $\deg_{x_1^T}(a) > 3$ ; and part a) of Lemma 3.2 implies  $a = 0$  in  $A_{3,d}$ . Lemma 6.1 completes the proof.

**c)** The required follows from part b) of Lemma 3.2, equality (A) of Lemma 5.1, and Lemma 6.1.

□

**Remark 6.4.** Denote by  $D_{\text{nil}}$  the *nilpotency degree* of  $A_{3,d}$ , i.e., the minimal  $s$  such that  $a_1 \cdots a_s = 0$  in  $A_{3,d}$  for all  $a_1, \dots, a_s \in \mathcal{M}_{\mathbb{F}}$ . It is not difficult to see that if  $d > 1$ , then the same estimations as for  $D_{\max}$  in Lemma 6.3 are also valid for  $D_{\text{nil}} - 1$ .

**Hypothesis 6.5.** If  $\text{char } \mathbb{F} = 3$  and  $d \geq 7$ , then  $D_{\max} = 2d + 7$ .

**Remark 6.6.** Let  $\text{char } \mathbb{F} = 3$  and  $d \geq 7$ . To prove Hypothesis 6.5 it is enough to show that

$$x_1^2 W_{23} x_1 W_{45} W_{67} \neq 0 \text{ in } A_{3,d}, \quad (15)$$

where  $W_{ij} = x_i^2 x_j^2 x_i x_j$ . This claim follows from part b) of Corollary 5.5 and Statement 8 of [5].

## References

- [1] H. Aslaksen, E.-C. Tan, C.-B. Zhu, *Invariant theory of special orthogonal groups*, Pac. J. Math. **168** (1995), No. 2, 207–215.
- [2] H. Aslaksen, V. Drensky, L. Sadikova, *Defining relations of invariants of two  $3 \times 3$  matrices*, J. Algebra **298** (2006), 41–57.
- [3] M. Domokos, S.G. Kuzmin, A.N. Zubkov, *Rings of matrix invariants in positive characteristic*, J. Pure Appl. Algebra **176** (2002), 61–80.
- [4] S. Donkin, *Invariants of several matrices*, Invent. Math. **110** (1992), 389–401.
- [5] A.A. Lopatin, *The algebra of invariants of  $3 \times 3$  matrices over a field of arbitrary characteristic*, Comm. Algebra **32** (2004), No. 7, 2863–2883.
- [6] A.A. Lopatin, *Relatively free algebras with the identity  $x^3 = 0$* , Comm. Algebra **33** (2005), No. 10, 3583–3605.

- [7] A.A. Lopatin, *On block partial linearizations of the pfaffian*, Linear Algebra Appl. **426/1** (2007), 109–129.
- [8] A.A. Lopatin, A.N. Zubkov, *Semi-invariants of mixed representations of quivers*, Transform. Groups **12** (2007), N2, 341–369.
- [9] A.A. Lopatin, *Invariants of quivers under the action of classical groups*, J. Algebra **321** (2009), 1079–1106.
- [10] A.A. Lopatin, *Indecomposable invariants of representations of quivers for dimension  $(2, \dots, 2)$  and maximal paths*, submitted, arXiv: 0704.2411.
- [11] A.A. Lopatin, *The Procesi–Razmyslov theorem for  $O(n)$ -invariants in prime characteristic*, arXiv: 0902.4266.
- [12] K. Nakamoto, *The structure of the invariant ring of two matrices of degree 3*, J. Pure Appl. Algebra **166** (2002), 125–148.
- [13] C. Procesi, *The invariant theory of  $n \times n$  matrices*, Adv. Math. **19** (1976), 306–381.
- [14] C. Procesi, *Computing with  $2 \times 2$  matrices*, J. Algebra **87** (1984), 342–359.
- [15] Yu.P. Razmyslov, *Trace identities of full matrix algebras over a field of characteristic 0*, Izv. Akad. Nauk SSSR Ser. Mat. **38** (1974), No. 4, 723–756 (Russian).
- [16] K.S. Sibirskii, *Algebraic invariants of a system of matrices*, Sibirsk. Mat. Zh. **9** (1968), No. 1, 152–164 (Russian).
- [17] A.N. Zubkov, *On a generalization of the Razmyslov–Procesi theorem*, Algebra and Logic **35** (1996), No. 4, 241–254.
- [18] A.N. Zubkov, *Invariants of an adjoint action of classical groups*, Algebra and Logic **38** (1999), No. 5, 299–318.
- [19] A.N. Zubkov, *Invariants of mixed representations of quivers II: Defining relations and applications*, J. Algebra Appl. **4** (2005), No. 3, 287–312.